QUESTION BANK
REAL ANALYSIS
P-102

1 mark questions

1. Define the Riemann-Stieltjes integral: $\int f(x)\, d\alpha(x)$.
2. Give the conditions for the existence of the Riemann-Stieltjes integral: $\int f(x)\, d\alpha(x)$ exists if...
3. Calculate the Riemann-Stieltjes integral: $\int x^2\, d\alpha(x)$ where $\alpha(x) = \sin(x)$.
4. Prove the linearity property of the Riemann-Stieltjes integral.
5. Evaluate $\int \cos(x)\, d|x|$ over the interval $[-\pi, \pi]$.
6. Differentiate the function $F(x) = \int_0^x e^t\, d\alpha(t)$.
7. State and prove the integration by parts formula for the Riemann-Stieltjes integral.
8. Show that if $\alpha(x)$ is continuous, then $\int f(x)\, d\alpha(x)$ is also continuous.
9. Define the pointwise convergence of a sequence of functions: $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in \text{domain}(f)$.
10. Prove that a uniformly convergent sequence of continuous functions has a continuous limit.
11. Determine whether the sequence of functions $f_n(x) = x^n$ on the interval $[0, 1]$ converges uniformly or not.
12. Calculate the limit of the sequence of functions $f_n(x) = n \sin(x/n)$ as n approaches infinity.
13. Show that if $f_n \to f$ uniformly, then $\int f_n(x)\, dx \to \int f(x)\, dx$ as $n \to \infty$.
14. Differentiate the function $f(x) = \sum_{n=1}^{\infty} x^n$ with respect to x.
15. Prove that the sequence of functions $f_n(x) = 1/(1 + nx)$ converges uniformly on $[0, 1]$.
16. Evaluate the limit of the series $\sum_{n=1}^{\infty} (-1)^n/n^2$.
17. Define the Lebesgue measure of a set A: $m(A) = ?$
18. Show that the Lebesgue measure of an interval $[a, b]$ is equal to its length: $m([a, b]) = ?$
19. Give an example of a non-measurable set on the real line.
20. Prove that the union of countably many Lebesgue measurable sets is also measurable.
21. Evaluate the Lebesgue integral: $\int_0^1 x^2\, d\lambda$, where λ is the Lebesgue measure.
22. Show that a Riemann integrable function on a closed interval is also Lebesgue integrable.
23. Prove that if $f(x)$ is non-negative and Lebesgue integrable, then $\int f(x)\, dx \geq 0$.
24. Discuss the relationship between outer measure and Lebesgue measure.
25. State the differentiation theorem for monotone functions.

26. Prove that a function of bounded variation is differentiable almost everywhere.
27. Define absolute continuity of functions and its relation to the Lebesgue integral.
28. Show that the space $C([a, b])$ of continuous functions on $[a, b]$ is a Banach space.
29. Evaluate the variation of the function $f(x) = x^2$ on the interval $[0, 1]$.
30. Define the Lp space, $1 \leq p < \infty$, and show that it is a Banach space.
31. Prove that the L^∞ norm is equal to the essential supremum norm.
32. Prove the Cauchy criterion for uniform convergence of functions.
33. Differentiate the function $f(x) = \ln|x|$ using the Riemann-Stieltjes integral with respect to the function $\alpha(x) = x^2$.
34. Evaluate $\int_0^\pi e^x \sin(x)\, dx$.
35. Show that the sequence of functions $f_n(x) = n^2x(1-x)^n$ converges uniformly on the interval $[0, 1]$.
36. Prove that the set of discontinuities of a Riemann-Stieltjes integrable function is of Lebesgue measure zero.
37. Differentiate the function $F(x) = \int_0^x e^t\, dt$ with respect to x.
38. Evaluate the limit of the sequence of functions $f_n(x) = n^2 \sin(x/n)$ as n approaches infinity.
39. Prove that if $f_n \to f$ uniformly, then $\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx$.

2/3 marks questions

1. State and prove the Mean Value Theorem for the Riemann-Stieltjes integral.
2. Find the Riemann-Stieltjes integral of $f(x) = x^2$ with respect to $\alpha(x) = e^x$ on the interval $[0, 1]$.
3. Prove that if $\alpha(x)$ is of bounded variation on $[a, b]$, then the set of points where $\alpha(x)$ is discontinuous is countable.
4. Differentiate the function $F(x) = \int_0^x \sin(t)\, d\alpha(t)$ with respect to x, where $\alpha(x) = x^2$.
5. Discuss the relation between the Riemann integral and the Riemann-Stieltjes integral.
6. Define pointwise convergence and uniform convergence of a sequence of functions.
7. State the Weierstrass M-Test for uniform convergence of a series of functions.

8. Differentiate the series representation of the function $f(x) = \ln(1+x)$ term-by-term.
9. Prove that if a sequence of functions $\{f_n\}$ is uniformly convergent, then it is also bounded.
10. Show that the series $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$ converges uniformly on $[0, 1]$.
11. Prove that the Lebesgue measure of the Cantor set is zero.
12. Evaluate the Lebesgue integral: $\int_0^1 x^2 \, d\lambda$, where λ is the Lebesgue measure.
13. Define a Lebesgue measurable function and prove that the pointwise limit of a sequence of measurable functions is measurable.
14. Prove that the set of rational numbers on the real line is not Lebesgue measurable.
15. Discuss the relationship between the Lebesgue integral and the Riemann-Stieltjes integral.
16. State the Lebesgue's differentiation theorem and its significance in real analysis.
17. Differentiate the function $f(x) = \int_0^x e^t \, dt$ with respect to x using the Fundamental Theorem of Calculus.
18. Prove that the space $C([a, b])$ of continuous functions on $[a, b]$ is a Banach space.
19. Define the space $L^p([a, b])$ for $1 \leq p < \infty$, and show that it is a Banach space.
20. Prove the Holder's inequality for two functions $f(x)$ and $g(x)$ with exponents p and q satisfying $1/p + 1/q = 1$.
21. Differentiate the function $F(x) = \int_0^{x^2} e^t \, dt$ using the chain rule and the Fundamental Theorem of Calculus.
22. Evaluate the limit of the sequence of functions $f_n(x) = n \sin(\pi x/n)$ as n approaches infinity.
23. Show that if $f_n \to f$ uniformly and $g_n \to g$ uniformly, then $f_n + g_n \to f + g$ uniformly.
24. Prove that a Riemann-Stieltjes integrable function on a closed interval is also Lebesgue integrable.
25. Evaluate $\int_0^\pi x \sin(x) \, dx$ using integration by parts.

6/7 marks questions

1. Prove the Riemann-Stieltjes integration by parts formula: $\int f(x)\, d\alpha(x) \cdot g(x) = f(x) \cdot g(x) \,|[a, b] - \int g(x)\, d\beta(x) \cdot f'(x)\, dx\, |[a, b]$, where $\alpha(x)$ and $\beta(x)$ are increasing functions and $f(x)$ is continuously differentiable on $[a, b]$.
2. Using the Riemann-Stieltjes integral, evaluate $\int_0^1 x^2\, d\alpha(x)$, where $\alpha(x)$ is the floor function.
3. Prove that if $\alpha(x)$ is of bounded variation on $[a, b]$, then the set of points where $\alpha(x)$ is discontinuous is countable.
4. Show that the Riemann-Stieltjes integral $\int f(x)\, d\alpha(x)$ exists for a function $f(x)$ that has a jump discontinuity at a point c and $\alpha(x)$ is continuous at c.
5. Differentiate the function $F(x) = \int_0^x t \sin(t)\, d\alpha(t)$ with respect to x, where $\alpha(x) = e^x$.
6. Prove that the uniform limit of a sequence of continuous functions is continuous.
7. Show that the sequence of functions $f_n(x) = nx/(1 + n^2x^2)$ converges uniformly on the interval $[0, 1]$.
8. Differentiate the series representation of the function $f(x) = \sin(x)$ term-by-term.
9. Prove that the series $\sum_{n=1}^{\infty} (x^n)/(1+x^n)$ converges uniformly on $[0, 1]$.
10. State and prove the Dini's theorem for uniform convergence of a sequence of functions.
11. Prove that every open interval is Lebesgue measurable and has the same Lebesgue measure as its length.
12. Evaluate the Lebesgue integral: $\int_0^1 x^3\, d\lambda$, where λ is the Lebesgue measure.
13. Define the space $L^p([a, b])$ for $1 \le p < \infty$ and show that it is a Banach space.
14. Prove that the Lebesgue integral of a non-negative Lebesgue measurable function $f(x)$ is always non-negative.
15. Show that if $f_n(x) \to f(x)$ pointwise for Lebesgue measurable functions on $[a, b]$, and there exists a Lebesgue integrable function $g(x)$ such that $|f_n(x)| \le g(x)$ for all n, then $f(x)$ is Lebesgue integrable.

16. Prove the Lebesgue's differentiation theorem: If $f \in L^1([a, b])$, then the function $F(x) = \int_a^x f(t)\, dt$ is differentiable almost everywhere on $[a, b]$ and $F'(x) = f(x)$ almost everywhere.
17. Evaluate the derivative of the function $F(x) = \int_0^x e^t\, dt$ using the Lebesgue differentiation theorem.
18. Define absolute continuity of functions and prove that an absolutely continuous function is uniformly continuous.
19. Show that the space $L^p([a, b])$ for $1 \leq p < \infty$ is a Banach space.
20. Prove Holder's inequality for two functions $f(x)$ and $g(x)$ with exponents p and q satisfying $1/p + 1/q = 1$.
21. Prove the convergence of the series $\sum_{n=1}^{\infty} (1 - 1/n)^n$.
22. Show that if $f_n \to f$ uniformly and $g_n \to g$ uniformly, then $f_n \cdot g_n \to f \cdot g$ uniformly.
23. Evaluate the limit of the sequence of functions $f_n(x) = (1 + x/n)^n$ as n approaches infinity.
24. Prove that if f_n is a sequence of Riemann-Stieltjes integrable functions on $[a, b]$ converging uniformly to f and α is an increasing function of bounded variation, then f is Riemann-Stieltjes integrable with respect to α.
25. Using the Lebesgue's differentiation theorem, find the derivative of the function $F(x) = \int_0^{x^2} e^t\, dt$.